4.6-4.7 first-order linear systems

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Recall: We can convert higher-order ODEs into a system of first-order ODEs

$$x^{(n)} + a_1 x^{(n-1)} + a_2 x^{(n-2)} + \cdots + a_{n-1}\dot{x} + a_n x = g(t).$$

$$\Rightarrow \quad \begin{aligned} x_1 &= x \\ x_2 &= \dot{x} \\ x_3 &= \ddot{x} \\ &\vdots \\ x_n &= x^{(n-1)} \end{aligned} \quad , \quad \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= -a_1 x_n - a_2 x_{n-1} - \cdots - a_{n-1} x_2 - a_n x_1 + g(t) \end{aligned}$$

$$\Rightarrow \quad \frac{dX}{dt} = AX + G(t), \quad \text{where}$$

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad A = \underbrace{\begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \\ -a_n & -a_{n-1} & \cdots & -a_2 & -a_1 \end{bmatrix}}_{\text{companion matrix}}, \quad G(t) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ g(t) \end{bmatrix}$$

Ex. $\ddot{x} + 4\dot{x} + 3x = \sin(t)$

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \sin(t) \end{bmatrix}$$

Solving linear systems: $\frac{dX}{dt} = A(t)X(t) + G(t)$

(in general, not just systems arising from higher-order ODEs)

$$X(t) = \underbrace{X_h(t)}_{\text{homogeneous solution}} + \underbrace{X_p(t)}_{\text{particular solution}}, \quad \text{where} \quad \begin{aligned} \dot{X}_h(t) &= A(t)X_h(t) \\ \dot{X}_p(t) &= A(t)X_p(t) + G(t). \end{aligned}$$

Let $\phi_1(t), \ldots, \phi_n(t)$ be linearly independent sol. to the homog eq^n.

Then we have fundamental matrix of solutions

$$\Phi(t) = [\phi_1(t) \cdots \phi_n(t)].$$

Note $\det(\Phi(t)) \neq 0$, because of linear ind.

$$X(t) = \Phi(t) \cdot \Phi^{-1}(t_0) \cdot X_0 + \Phi(t) \int_{t_0}^{t} \Phi^{-1}(s) G(s) ds$$

where $X(t_0) = X_0$.

Constant coefficients

$$\frac{dX}{dt} = AX, \quad \text{where} \quad A = (a_{ij}), \quad a_{ij} \in \mathbb{R}.$$

By Picard iteration, $X(t) = e^{At} X_0$, where

$$e^{At} = I + At + A^2 \cdot \frac{t^2}{2!} + A^3 + \frac{t^3}{3!} + \cdots = \sum_{k=0}^{\infty} A^k \cdot \frac{t^k}{k!},$$

and $X_0 \in \mathbb{R}^{n \times 1}$.

Ex. 4.9. Suppose $A = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}$, $X_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$

$$e^{At} = \begin{pmatrix} e^{a_{11}t} & 0 \\ 0 & e^{a_{22}t} \end{pmatrix}. \quad \text{Then } X(t) = e^{At} X_0 = \begin{bmatrix} e^{a_{11}t} x_0 \\ e^{a_{22}t} x_0 \end{bmatrix} = x_0 e^{a_{11}t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y_0 e^{a_{22}t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

In general, if can find eigendecomposition of A with n linearly ind. eigenvectors can find solutions to ODE.

Case 1: Let $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ be eigenvalues of A with linearly ind. eigenvectors V_i.

Then for $\dot{X}(t) = A(t) X(t)$ we have solutions

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$$X(t) = \sum_{i=1}^{n} c_i V_i e^{\lambda_i t}$$

if all eigenvalues have neg. real part, this $\to 0$.

Consider the case without n linearly ind. eigenvectors for $n=2$.

Case 2: Let $n=2$. $\lambda_1 = \lambda_2 = \lambda$ but only one eigenvector V_1.

Then $X(t) = c_1 V_1 e^{\lambda t} + c_2 \left[V_1 t e^{\lambda t} + P e^{\lambda t} \right]$

where P is a generalized eigenvector $(A - \lambda I) P = V_1$.

Case 3: $\lambda_1, \lambda_2 \in \mathbb{C}$ and $\lambda_1 = \bar{\lambda}_2$, $\lambda_1 = a + ib$, $\lambda_2 = a - ib$, $b \neq 0$.

Then $$X(t) = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} e^{at} \sin(bt) + \begin{pmatrix} \alpha_3 \\ \alpha_4 \end{pmatrix} e^{at} \cos(bt).$$

Note that there are actually only two degrees of freedom c_1, c_2, despite $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ appearing above. Can use Euler's equation.